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On Fermi-like Quantisation of Classical Mechanics

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Abstract

In this work we generalise some previously obtained results concerning the quantisation of classical finite models according to the symmetric (Fermi-like) scheme of quantisation. We consider models whose dynamics is defined through some non-singular Lie bracket and show that we can make the dynamics with any prescribed bracket relations, as defined by a certain type of *non-singular symmetric brackets*, coexist. The quantisation scheme established is: (a) defined up to an arbitrary factor and, (b) sensitive to the addition of total time derivatives to the corresponding Lagrangian. Both unconstrained and constrained models are considered.

1. *Introduction*

During the past few years a certain number of works have been devoted to extend the usual (Dirac's) quantisation scheme in order to treat both Bose-like and Fermi-like systems in a unified way (Droz-Vincent, 1966; Franke & Kálnay, 1970; Kálnay & Ruggeri, 1972). Specifically, in a recent work, Kálnay & Ruggeri (1972) have presented a classical model of coupled harmonic oscillators that, when quantised via the symmetric (Fermi-like, rule of quantisation, produced just the usual non-relativistic system of second-quantised fermions. This model showed the following peculiarities: (a) the quantisation rule was defined up to an arbitrary factor, (b) the symmetric bracket relations were sensitive to the addition of a total time derivative to the model's Lagrangian, and (c) it was possible to make some prescribed symmetric bracket relations with a given set of dynamical equations coexist. The purpose of this work is to show that these properties are not specific of the mentioned model but, on the contrary, that they are present in the wider class of models characterised by: (a) a non-singular skew-symmetric bracket which define the dynamics in the conventional way, and (b) a non-singular symmetric bracket which prescribes the anticommutator relations of the quantised model.

It is also the purpose of this work to introduce a slightly generalised version of the plus Dirac bracket and to show that it is the natural symmetric structure associated with a class of models for which there exists *a priori* constraints between the coordinates.

2. *Notations and Conventions*

In Section 3 we are going to consider classical finite models whose phase-space coordinates we denote by α^μ, $\mu = 1, \ldots, ZN$. Later on this space will be stretched by introducing canonically conjugated momenta which will be denoted by π_μ.

The sum convention for any kind of indices, as well as the abbreviations $\partial_\mu \equiv \partial/\partial\alpha^\mu$, $\partial_{\mu\nu} \equiv \partial^2/\partial\alpha^\mu\partial\alpha^\nu$ and $\partial^\mu \equiv \partial/\partial\pi_\mu$, will be used systematically throughout this work.

Brackets. The symbol $\{\ ,\ \}_+$ denote the plus Poisson bracket (Droz-Vincent, 1966):

$$\{F, G\}_+ = \partial_\mu F\, \partial^\mu G + \partial_\mu G\, \partial^\mu F, \tag{2.1}$$

which is defined in the stretched phase space spanned on by $\{\alpha, \pi\}$, whereas $\{\ ,\ \}_\mp{}^\Gamma$ is used for the brackets

$$\{F, G\}_\mp{}^\Gamma = \Gamma_\mp{}^{\mu\nu}\, \partial_\mu F\, \partial_\nu G, \tag{2.2}$$

which are defined in the original phase space. Here $\|\Gamma_+{}^{\mu\nu}\|$ is a symmetric matrix and $\|\Gamma_-{}^{\mu\nu}\|$ a skew-symmetric one.

On the other hand, $\{\ ,\ \}_+{}^*$ is the plus Dirac bracket defined by (Franke & Kálnay, 1970)

$$\{F, G\}_+{}^* = \{F, G\}_+ - \{F, \theta_a\}_+ C_+{}^{ab}\{\theta_b, G\}_+. \tag{2.3}$$

Finally, $\{\ ,\ \}_+{}^{\Gamma^*}$ is the generalised Dirac bracket introduced in Section 4 and and $[\ ,\ ]_+$ denotes the anticommutator.

Constraints. In Section 4 a constraint will be called plus first-class constraint (respectively plus second-class constraint) when it is first class (respectively second class) with respect to $\{\ ,\ \}_+$. When classified in the same way, but according to the bracket $\{\ ,\ \}_+{}^\Gamma$, they are called Γ_+-first-class or Γ_+-second-class constraints.

3. *Lagrangian Formulation of Symmetric Quantisation*

We consider the symmetric quantisation problem for a classical system whose dynamical equations are of the form

$$\dot{\alpha}^\mu = \Gamma_-{}^{\mu\nu}(\alpha)\, \partial_\mu H = \{\alpha^\mu, H\}_-{}^\Gamma. \tag{3.1}$$

Here the $\Gamma_-{}^{\mu\nu}$ define a skew-symmetric non-singular matrix which allows us to construct in the usual way the Lie bracket $\{\ ,\ \}_-{}^\Gamma$. These dynamical equations are in fact the Euler–Lagrange equations corresponding to the family of Lagrangians (Ruggeri, 1973)†

$$L_G = f_\sigma(\alpha)\dot{\alpha}^\sigma - H(\alpha) + \frac{dG}{dt}. \tag{3.2}$$

The f_σ are such that $\|\Gamma^-{}_{\mu\nu}\|$, the matrix inverse to $\|\Gamma_-{}^{\mu\nu}\|$, can be obtained as

$$\Gamma^-{}_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu. \tag{3.3}$$

G is an arbitrary 'gauge' function which is in fact irrelevant for both: (a) the dynamical equations and (b) the Bose-like (skew-symmetric) quantisation problem. *This however is not true as regards the Fermi-like quantisation scheme.* To see this, notice that when constructing the canonical formalism from L_G the following $2N$ primary constraints appear in the stretched phase space

$$\chi_\mu \equiv \pi_\mu - f_\mu - \partial_\mu G \approx 0. \tag{3.4}$$

These constraints, which are the only ones in this case, are always *minus* second-class constraints irrespective of G. Their plus character however strongly depends on G because

$$\{\chi_\mu, \chi_\nu\}_+ = -(\partial_\mu f_\nu + \partial_\nu f_\mu) - 2\partial_{\mu\nu}G. \tag{3.5}$$

Let us now consider any non-singular (sufficiently well behaved) symmetric matrix $\Gamma_+{}^{\mu\nu}(\alpha)$. It is always possible to choose G in such a way that equation (3.5) becomes

$$\{\chi_\mu, \chi_\nu\}_+ = -\xi_+\Gamma_{\mu\nu}{}^+, \tag{3.6}$$

where $\|\Gamma_{\mu\nu}{}^+\|$ is the matrix inverse to $\|\Gamma_+{}^{\mu\nu}\|$ and ξ_+ is any non-zero complex constant. This means that we can always choose G so that the constraints (3.4) are simultaneously minus second class and plus second class. Any further addition of non-linear G's to the Lagrangian (3.2) will change the plus character of the constrains and, consequently will change also the symmetric bracket relations between the α's which we now consider.

The Fermi-like quantisation scheme proceeds through the construction of the plus Dirac bracket (Franke & Kálnay, 1970) which is defined as

$$\{F, J\}_+{}^* = \{F, J\}_+ - \{F, \theta_a{}^+\}_+ C_+{}^{ab}\{\theta_b{}^+, J\}_+. \tag{3.7}$$

Here $\{\theta^+\} = \{\chi\}$ and then, by equation (3.6): $\|C_+{}^{ab}\| = \xi_+{}^{-1}\|\Gamma_+{}^{\mu\nu}\|$. If F and J are functions of α but not of π, we have

$$\{F, J\}_+{}^* = \xi_+{}^{-1}\{F, \chi_\mu\}_+\Gamma_+{}^{\mu\nu}\{\chi_\nu, G\}_+ \tag{3.8}$$

or

$$\xi_+\{F, J\}_+{}^* = \Gamma_+{}^{\mu\nu}\,\partial_\mu F\,\partial_\nu J = \{F, J\}_+{}^\Gamma. \tag{3.9}$$

† For this $\{\alpha\}$ must be considered as a configuration space.

It follows then that the quantisation rule

$$\xi_+\{\ ,\ \}_+{}^* \to [\ ,\ ]_+ \tag{3.10}$$

leads here to

$$[\hat{\alpha}^\mu, \hat{\alpha}^\nu]_+ = O\Gamma_+{}^{\mu\nu}(\hat{\alpha}), \tag{3.11}$$

where $\hat{\alpha}^\mu$ is the quantum counterpart of α^μ and O is an ordering prescription chosen when quantising. The presence of the arbitrary factor ξ_+ in the correspondence (3.10) has been noted previously (Kálnay & Ruggeri, 1972).

Thus the Fermi-like quantisation according to any prescribed non-singular symmetric bracket can be made to coexist with any set of dynamical equations of the type described by equations (3.1) and (3.3). This is a generalisation of the result of a previous work where a model of coupled oscillators was quantised according to a particular symmetric bracket (Kálnay & Ruggeri, 1972). The method followed in that work was the one just described above.

4. *Constrained Models*

An interesting and natural extension of the results of the previous section is afforded by the case in which we have a certain number, say N_1, of *a priori* constraints $\phi_m(\alpha)$

$$\phi_m \approx 0 \qquad m = 1, \ldots, N_1. \tag{4.1}$$

Imposing the constraints as external conditions on the variational problem for the Lagrangian (3.2) we find the following dynamical equations (Ruggeri, 1973)

$$\dot{\alpha}^\mu = \Gamma_-{}^{\mu\nu}(\partial_\nu H + u^i\, \partial_\nu \phi_i). \tag{4.2}$$

As is usual N_1 additional variables, the u's, have appeared which are to be determined through the consistence equations $\dot{\phi}_i \approx 0$. These equations generally also produce new constraints which we again denote by ϕ_m, but now $m = N_1 + 1, \ldots, N_c$.

For the symmetric quantisation of the *a priori* constrained models it is convenient to consider the canonical formulation of equation (4.2). This formulation proceeds exactly as usual from equations (3.2) and (4.1). Now, however, the following stretched set of $2N + N_c$ constraints is obtained

$$\{\phi_m, \chi_\mu; m = 1, \ldots, N_c; \mu = 1, 2, \ldots, 2N\}. \tag{4.3}$$

Here the χ_μ are given by equation (3.4). We must now search for the plus character of these constraints. This is not *a priori* evident because the ϕ constraints do not have a null plus Poisson bracket with the χ constraints. To find an effective classification we substitute the ϕ's by the new set

$$\Phi_m{}^+ \equiv \phi_m + \xi_+{}^{-1}\chi_\lambda \Gamma_+{}^{\lambda\nu}\, \partial_\nu \phi_m \approx \phi_m \approx 0 \qquad m = 1, \ldots, N_c. \tag{4.4}$$

(cf. Ruggeri, 1973). It follows from equations (3.6) and (4.4) that:

(a)
$$\{\Phi_m{}^+, \chi_\mu\}_+ \approx \{\phi_m, \chi_\mu\}_+ + \xi_+{}^{-1}\{\chi_\lambda, \chi_\mu\}_+\Gamma_+{}^{\lambda\nu}\,\partial_\nu\phi_m$$
$$= \partial_\mu\phi_m - \Gamma^+{}_{\lambda\mu}\Gamma_+{}^{\lambda\nu}\,\partial_\nu\phi_m,$$

i.e.

$$\{\Phi_m{}^+, \chi_\mu\}_+ \approx 0, \tag{4.5}$$

and also

(b)
$$\{\Phi_m{}^+, \Phi_n{}^+\}_+ \approx \xi_+{}^{-1}\{\phi_m, \chi_\rho\}_+\Gamma_+{}^{\rho\nu}\,\partial_\nu\phi_n + \xi_+{}^{-1}\{\chi_\lambda, \phi_n\}_+\Gamma_+{}^{\lambda\mu}\,\partial_\mu\phi_m$$
$$+ \xi_+{}^{-2}\{\chi_\lambda, \chi_\rho\}_+\Gamma_+{}^{\lambda\mu}\,\partial_\mu\phi_m\Gamma_+{}^{\rho\nu}\,\partial_\nu\phi_n$$
$$= \xi_+{}^{-1}\Gamma_+{}^{\mu\lambda}\,\partial_\mu\phi_m\,\partial_\lambda\phi_n,$$

or

$$\{\Phi_m{}^+, \Phi_n{}^+\}_+ \approx \xi_+{}^{-1}\{\phi_m, \phi_n\}_+{}^\Gamma. \tag{4.6}$$

This way the set $\{\Phi_m{}^+\}$ has been classified according to the Γ_+-character of the set $\{\phi_m\}$: if ϕ is Γ_+-first-class (respectively Γ_+-second-class) then Φ^+ is plus first-class (respectively plus second-class).

Let us suppose now that from $\{\phi\}$ we can sort out an irreducible set of Γ_+-second-class constraints. Call them $\theta_a{}^+, a = 1, 2, \ldots, N_\theta$, with $N_\theta \leqslant N_c$. There there exists $\|C_{\Gamma_+}{}^{ab}\|$ such that

$$C_{\Gamma_+}{}^{ab}\{\theta_b{}^+, \theta_c{}^+\}_+{}^\Gamma = \delta_a{}^c. \tag{4.7}$$

Let us construct $\{\Theta_a{}^+\}$ according to equation (4.4):

$$\Theta_a{}^+ = \theta_a{}^+ + \xi_+{}^{-1}\chi_\lambda\Gamma_+{}^{\lambda\nu}\,\partial_\nu\theta_a{}^+. \tag{4.8}$$

We then argue that the set

$$\eta_u{}^+ = \delta_u{}^\mu\chi_\mu + \delta^a_{u-2N}\Theta_a{}^+, \qquad u = 1, 2, \ldots, 2N, \ldots, ZN + N_\theta, \tag{4.9}$$

is an irreducible set of plus second-class constraints. In fact the $(2N + N_\theta) \times (2N + N_\theta)$ matrix

$$\|C_+{}^{uv}\| = \begin{Vmatrix} -\xi_+{}^{-1}\Gamma_+ & 0 \\ 0 & \xi C_{\Gamma_+} \end{Vmatrix} \tag{4.10}$$

satisfies

$$C_+{}^{uv}\{\eta_v{}^+, \eta_w{}^+\}_+ = \delta_w{}^u, \qquad u, v, w = 1, 2, \ldots, 2N + N_\theta. \tag{4.11}$$

We are now able to calculate the symmetric bracket relations which follow from the plus Dirac bracket constructed with the constraints (4.9) and the matrix (4.10). This bracket is defined as

$$\{F, G\}_+^* = \{F, G\}_+ - \{F, \eta_u^+\}_+ C_+^{uv} \{\eta_v^+, G\}_+. \tag{4.12}$$

For two functions F and G which depend only on α we have, according to equations (4.8) and (4.9),

$$\{F, G\}_+ = 0, \tag{4.13a}$$

$$\{F, \eta_u^+\}_+ = \delta_u^\mu\, \partial_\mu F + \xi_+^{-1} \delta_{u-2N}^a \{F, \theta_a^+\}_+^\Gamma, \qquad \text{etc.} \tag{4.13b}$$

Substituting these into equation (4.12) we obtain

$$\xi_+ \{F, G\}_+^* = \Gamma_+^{\mu\nu}\, \partial_\mu F\, \partial_\nu G - \{F, \theta_a^+\}_+^\Gamma C_{\Gamma_+}^{ab} \{\theta_b^+, G\}_+^\Gamma \tag{4.14a}$$

or, in a condensed form,

$$\xi_+ \{F, G\}_+^* = \{F, G\}_+^{\Gamma *}, \tag{4.14b}$$

where the bracket $\{\ ,\ \}_+^{\Gamma *}$ has been constructed from $\{\ ,\ \}_+^\Gamma$ in the same way as $\{\ ,\ \}_+^*$ was constructed from $\{\ ,\ \}_+$ (see equation (2.3)).

We now reach the following conclusion: For any given constrained model whose dynamics is governed by equations (4.2) with a non-singular skew-symmetric matrix $\|\Gamma_-^{\mu\nu}\|$, a *symmetric* quantisation scheme can be set up. The procedure, which is completely analogous to the one followed in the skew-symmetric case (Ruggeri, 1973), leads us to a generalised version of the symmetric Dirac bracket.†

5. *Summary*

It has been shown in this work that, for at least a certain subset of classical dynamical models, a quantisation scheme of the symmetric type can be set up *irrespective of the specific form of the dynamical equations.* This means, in principle, we can quantise as a Fermi-like system any classical dynamical model of the above-defined subset. This in turn implies a certain degree of compatibility between different algebraic structures, which may be relevant when trying to construct classical analogues for generalised quantum systems. We refer, for instance, to Green parasystems for which, besides the skew-symmetric (Lie) structure, a symmetric one is needed (see, in this context, Kálnay, 1972).

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† It is worthwhile to point out here that the bracket $\{\ ,\}_+^{\Gamma *}$ can be defined by the same procedure followed by Bergmann & Goldberg (1955) and Mukunda & Sudarshan (1968) to define the skew-symmetric Dirac bracket, and by Franke & Kálnay (1970) to define the symmetric Dirac bracket. Moreover, the whole quantisation scheme constructed in the latter reference can be formulated entirely with the bracket $\{\ ,\}_+^{\Gamma *}$.

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